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Integrability conditions for shear-free motion in general relativity with applications to perfect fluids

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Abstract. In this paper we obtain the equations of shear-free motion of particles in a gravitational field in a form suitable for easy derivation of the corresponding integrability conditions. The results generalize those of Pirani and Williams in a similar study of rigid motions. We apply our results to the dynamics of a perfect fluid.

1. Introduction

There is currently some renewed interest in shear-free motion in the context of general relativity. Following thermodynamic considerations Treciokas and Ellis (1971) have obtained some interesting results on shear-free perfect fluids with certain specified equations of state.

Pirani and Williams (1962) derived integrability conditions for rigid motions. They found that the rigidity condition in general relativity associated with it a space-like three-space which is just the quotient space of space–time over the world lines. We find that this result may be extended to the case of shear-free motion where the quotient space is obtained not from space–time but from another space conformally related to it.

In § 2 we consider the equations for shear-free motion and obtain a number of lemmas which follow from the shear-free requirement. Integrability conditions for shear-free motion are derived in § 3 and we apply our results to perfect fluids in § 4 obtaining explicitly a set of 'selected variables'. There is a brief discussion of the results in § 5.

2. The equations of shear-free motion

The covariant derivative of the four-velocity vector u_a associated with a congruence of time-like curves may be decomposed in the usual way as follows (Ehlers and Kundt 1962)

$$u_{a\|b} \equiv \nabla_b u_a = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \alpha_a u_b \tag{2.1}$$

where

$$\omega_{ab} = u_{[a\|b]} + \alpha_{[a}u_{b]}, \qquad \sigma_{ab} = u_{(a\|b)} + \alpha_{(a}u_{b)} - \tfrac{1}{3}\theta h_{ab},$$
$$h_{ab} = g_{ab} + u_a u_b, \qquad \alpha_a = u_{a\|b}u^b, \qquad \theta = u^a{}_{\|a} \qquad \text{and} \qquad u_a u^a = -1. \tag{2.2}$$

Here Latin indices range and sum over 1, 2, 3, 4. The motion of a continuous medium or a system of particles is said to be shear-free if and only if for its world lines the velocity of shear σ_{ab} vanishes:

$$\sigma_{ab} = 0. \tag{2.3}$$

If in addition the velocity of expansion θ vanishes, the motion is said to be rigid:

$$\sigma_{ab} = 0 = \theta. \tag{2.4}$$

Pirani and Williams (1962) showed that the equations (2.4) are equivalent to the requirement that the projection tensor h_{ab} be Lie transferred along the world lines:

$$\underset{u}{\pounds} h_{ab} = 0, \tag{2.5}$$

where the Lie derivative of an arbitrary tensor $X_{ab}{}^{c}$ over a vector field v^a is defined as

$$\underset{v}{\pounds} X_{ab}{}^{c} \equiv X_{ab}{}^{c}{}_{\|d} v^d + v^d{}_{\|a} X_{db}{}^{c} + v^d{}_{\|b} X_{ad}{}^{c} - v^c{}_{\|d} X_{ab}{}^{d}. \tag{2.6}$$

For other formulae on Lie derivatives see for example Yano (1955). In this paper all Lie differentiation will be over the velocity vector u^a and in what follows we simply write £ for $\pounds_u$.

We may easily show that a motion is shear free if and only if

$$\pounds h_{ab} = \tfrac{2}{3}\theta h_{ab}. \tag{2.7}$$

This may be written in the form

$$\pounds(\mathrm{e}^{2\phi} h_{ab}) = 0, \qquad \pounds\phi = -\tfrac{1}{3}\theta \tag{2.8}$$

where ϕ is a scalar. We shall see that this form is particularly suited to the derivation of the corresponding integrability conditions.

Under the shear-free conditions it may be shown that

$$\pounds\alpha_b = \perp\dot{\alpha}_b - \omega_{bc}\alpha^c + \tfrac{1}{3}\theta\alpha_b \tag{2.9}$$

and

$$\pounds\omega_{ab} = \perp\dot{\omega}_{ab} + \tfrac{2}{3}\theta\omega_{ab} \tag{2.10}$$

where the tensor $\dot{X}_{ab}{}^{c} \equiv X_{ab}{}^{c}{}_{\|d} u^d$ and $\perp X_{ab}{}^{c} \equiv h^p_a h^q_b h^c_r X_{pq}{}^{r}$.

It is always possible to adapt the coordinate system in such a way that the proper time s becomes the new coordinate x^4 so that

$$u^a \equiv \frac{\partial x^a}{\partial s} \triangleq \delta^a_4, \qquad u_a \triangleq g_{a4}, \qquad u_4 \triangleq g_{44} \triangleq -1 \tag{2.11}$$

where the symbol $\triangleq$ indicates that the equation is valid in the adapted coordinate system but need not be so in general. For an arbitrary tensor $X^{cd\ldots}_{ab\ldots}$,

$$\perp X^{cd\ldots}_{ab..4.} \triangleq 0$$

and

$$\pounds X^{cd\ldots}_{ab\ldots} \triangleq \frac{\partial}{\partial x^4} X^{cd\ldots}_{ab\ldots} \tag{2.12}$$

In this coordinate system we therefore have the following theorem.

Theorem. A necessary and sufficient condition for a space–time to admit a shear-free motion is that there exists an adapted coordinate system with respect to which the components $h_{\alpha\beta}$ satisfy the equations

$$\frac{\partial}{\partial x^4}(e^{2\phi}h_{\alpha\beta}) \doteq 0, \qquad \text{with } \frac{\partial \phi}{\partial x^4} \doteq -\tfrac{1}{3}\theta, \tag{2.13}$$

where Greek indices range over 1, 2, 3.

We see from (2.13) that, provided ϕ satisfies the second equation, $e^{2\phi}h_{\alpha\beta}$ is a non-singular matrix which depends only on the three space coordinates x^α and hence we may regard it as the metric of a three-dimensional riemannian space which is just the quotient space of the four-dimensional space with metric $e^{2\phi}g_{ab}$ over the world lines associated with u^a. We may define the metric $\overset{*}{g}_{\alpha\beta}$ of the three-space $\overset{*}{V}$ by

$$\overset{*}{g}_{\alpha\beta} \doteq e^{2\phi}h_{\alpha\beta}, \qquad \text{with } \overset{*}{g}{}^{\alpha\beta} \doteq e^{-2\phi}g^{\alpha\beta} \tag{2.14}$$

so that $\overset{*}{g}_{\alpha\beta}\overset{*}{g}{}^{\beta\gamma} \doteq \delta^\gamma_\alpha$. The corresponding riemannian connection is given by

$$\overset{*}{\Gamma}{}^\gamma_{\alpha\beta} \doteq \tilde{\Gamma}^\gamma_{\alpha\beta} + (\delta^\gamma_\alpha \phi_{|\beta} + \delta^\gamma_\beta \phi_{|\alpha} - h_{\alpha\beta}h^{\gamma\delta}\phi_{|\delta}) \tag{2.15}$$

where $\tilde{\Gamma}^\gamma_{\alpha\beta}$ is the riemannian connection computed from the tensor $h_{\alpha\beta}$ and $\phi_{|\alpha} \equiv \partial\phi/\partial x^\alpha$.

We prove the following lemmas with the aid of the adapted coordinate system and the quotient space $\overset{*}{V}$ of the shear-free space–time.

Lemma 1. Let X_a be any vector field in a shear-free space–time V such that

$$X_a u^a = 0 \qquad \text{and} \qquad \pounds X_a = 0. \tag{2.16}$$

Let $\overset{*}{\nabla}_\beta$ denote the covariant derivative operator in $\overset{*}{V}$. Then in the adapted coordinate system

$$\overset{*}{\nabla}_\beta X_\alpha \doteq \perp\Delta_\beta X_\alpha \tag{2.17}$$

where Δ_β is the covariant differentiation operator defined by

$$\Delta_\beta X_\alpha \triangleq \nabla_\beta X_\alpha - (\delta^\gamma_\alpha \perp\phi_{|\beta} + \delta^\gamma_\beta \perp\phi_{|\alpha} - h_{\alpha\beta}h^{\gamma\delta}\perp\phi_{|\delta})X_\gamma \tag{2.18}$$

and $\perp\phi_{|\alpha} \equiv h^c_\alpha \phi_{|c}$.

Proof. In the adapted coordinate system the equations (2.16) become

$$X_4 \doteq 0, \qquad \frac{\partial}{\partial x^4}X_\alpha \doteq 0$$

and hence X_α is a vector field in $\overset{*}{V}$. From its definition and (2.15)

$$\overset{*}{\nabla}_\beta X_\alpha \doteq \tilde{\nabla}_\beta X_\alpha - (\delta^\gamma_\alpha \phi_{|\beta} + \delta^\gamma_\beta \phi_{|\alpha} - h_{\alpha\beta}h^{\gamma\delta}\phi_{|\delta})X_\gamma$$

where

$$\tilde{\nabla}_\beta X_\alpha \triangleq \partial_\beta X_\alpha - \tilde{\Gamma}^\gamma_{\beta\alpha}X_\gamma.$$

We may also show that

$$\perp\nabla_\beta X_\alpha \doteq \tilde{\nabla}_\beta X_\alpha - \tfrac{1}{3}\theta(\delta^\gamma_\alpha u_\beta + \delta^\gamma_\beta u_\alpha - h_{\alpha\beta}h^{\gamma\delta}u_\delta)X_\gamma.$$

The lemma follows from these equations since $\perp\phi_{|\alpha} = \phi_{|\alpha} - \tfrac{1}{3}\theta u_\alpha$.

Lemma 2. If any vector X_a satisfies the conditions of lemma 1 then

$$\pounds\perp\Delta_b\perp X_a = 0, \tag{2.19}$$

where the generalized covariant derivative Δ_b is defined by

$$\Delta_b X_a \equiv \nabla_b X_a - \Phi^c_{ba} X_c$$

and

$$\Phi^c_{ba} \equiv h^c_a \perp \phi_{|b} + h^c_b \perp \phi_{|a} - h_{ab} h^{cd} \phi_{|d}. \qquad (2.20)$$

Proof. Since X_α is a vector field in $\overset{*}{V}$ and $\overset{*}{\nabla}_\beta$ is the covariant derivative operator in $\overset{*}{V}$, $\overset{*}{\nabla}_\beta X_\alpha$ is a tensor field in $\overset{*}{V}$ and hence

$$\frac{\partial}{\partial x^4} \overset{*}{\nabla}_\beta X_\alpha \doteq 0.$$

It follows from lemma 1 and (2.12) that

$$£ \perp \Delta_\beta X_\alpha \doteq 0$$

so that

$$£ \perp \Delta_b X_a \doteq 0.$$

Since this is a tensor equation valid in one coordinate system of V it is valid in every coordinate system and the lemma follows.

Similar results may be obtained for a scalar field χ for which $£\chi = 0$. Furthermore one may show that when the conditions of lemma 1 are satisfied the following extensions of lemmas 1 and 2 are true:

$$\overset{*}{\nabla}_\gamma \overset{*}{\nabla}_\beta X_\alpha \doteq \perp \Delta_\gamma \perp \Delta_\beta X_\alpha \qquad (2.21)$$

and

$$£ \perp \Delta_c \perp \Delta_b X_a = 0 \qquad \text{etc.} \qquad (2.22)$$

More generally one may prove the following lemma.

Lemma 3. Let V be a shear-free space–time and let $X^{b_1 \ldots b_q}_{a_1 \ldots a_p}$ be a tensor field in V which is orthogonal to u^a on each contravariant and covariant index. If in addition

$$£ X^{b_1 \ldots b_q}_{a_1 \ldots a_p} = 0,$$

then

$$\perp £ \perp \Delta_c X^{b_1 \ldots b_q}_{a_1 \ldots a_p} = 0, \qquad (2.23)$$

where differentiation with Δ_c is defined by an obvious extension of the definition in (2.20).

3. Integrability conditions for shear-free motion

We now set out to find compatibility conditions for a given riemannian space–time to admit a shear-free motion. In doing this we assume that the metric tensor g_{ab}, its affine connection $\{^{\,c}_{ab}\}$, Riemann tensor R_{abcd} and all its derivatives are given. The condition for shear-free motion

$$u_{(a\|b)} + \alpha_{(a} u_{b)} - \tfrac{1}{3}\theta h_{ab} = 0 \qquad (3.1)$$

is then an equation involving u_a and its first covariant or partial derivatives. We therefore search for a set of variables to be called 'selected variables', obtained from u_a and its derivatives, whose first covariant or partial derivatives may be expressed algebraically in terms of the chosen variables. For this process we need to derive and make use of the integrability conditions for (3.1).

Since there are only five independent equations in (3.1), it is not possible to solve algebraically for the twelve first covariant derivatives $u_{a\|b}$ in terms of u_a and the geometric quantities g_{ab}, R_{abcd}, $R_{abcd\|e}$, etc. We may adjoin to the variable u_a, the first covariant derivatives ω_{ab}, α_a and θ. We then set out to obtain integrability conditions for this augmented set of variables namely u_a, ω_{ab}, α_a and θ. Referring to these variables for the moment as our selected variables we now have ten selected variables.

From equation (2.1) we see that the first covariant derivatives of u_a are already expressed algebraically in terms of the selected variables. By differentiating covariantly the relation $\alpha_b = u_{b\|c}u^c$ and using the Ricci identity together with the shear-free condition we obtain

$$\begin{aligned}\alpha_{b\|c} = {} & \dot{\omega}_{bc} + \tfrac{1}{3}\dot{\theta}h_{bc} - \dot{\alpha}_b u_c + \omega_b{}^k\omega_{kc} - \omega_{bk}\alpha^k u_c - \alpha_b\alpha_c \\ & + \tfrac{2}{3}\theta\omega_{bc} + \tfrac{1}{9}\theta^2 h_{bc} + \tfrac{1}{3}\theta u_b\alpha_c - R_{bsck}u^s u^k.\end{aligned} \tag{3.2}$$

By a similar procedure for ω_{ab} we obtain

$$\begin{aligned}\omega_{ab\|c} = {} & -\dot{\omega}_{ab}u_c - \tfrac{2}{3}\perp\theta_{\|[a}h_{b]c} - \omega_{ab}\alpha_c + 2\alpha_{[a}\omega_{b]c} - 2u_{[a}\omega_{b]k}\omega^k{}_{.c} \\ & - \tfrac{2}{3}\theta u_{[a}\omega_{b]c} - \perp R_{abck}u^k\end{aligned} \tag{3.3}$$

The equation (3.2) contains on the right-hand side the terms $\dot{\omega}_{ab}$, $\dot{\theta}$, $\dot{\alpha}_b$ while (3.3) has $\dot{\omega}_{ab}$ and $\perp\theta_{\|a}$. These new terms are derivatives of our selected variables and hence must be eliminated from the right-hand side of the equations. To do this we use the integrability conditions for the equations (3.2) and (3.3).

One may show that the integrability conditions for (3.2) are identically satisfied while those of (3.3) are not. To obtain these, instead of direct computation which is tedious, we proceed as follows.

Let V be a shear-free space–time and let X_a be an arbitrary covariant vector field lying in the rest space of the world lines in V. In addition, let X_a be Lie transferred along each world line. The equation (2.22) $\pounds\perp\Delta_c\perp\Delta_b X_a = 0$ follows. Our first integrability conditions are obtained by antisymmetrizing the indices b and c in this equation which on simplification reduces to

$$X_d\pounds\perp\{\tfrac{1}{2}R^d{}_{acb} + u_{a\|[b}u^d{}_{\|c]} - u^d{}_{\|a}u_{[b\|c]} + \nabla_{[b}\Phi^d_{c]a} + \Phi^e_{a[c}\Phi^d_{b]e}\} = 0. \tag{3.4}$$

Since X_d is an arbitrary vector in the infinitesimal three-space orthogonal to u_a, we conclude that

$$\perp\pounds\perp\{-\tfrac{1}{2}R^d{}_{abc} + u_{a\|[b}u^d{}_{\|c]} - u^d{}_{\|a}u_{[b\|c]} + \nabla_{[b}\Phi^d_{c]a} - \Phi^e_{a[b}\Phi^d_{c]e}\} = 0.$$

This may be put in the convenient form

$$\pounds(e^{2\phi}K_{abcd}) = 0 \tag{3.5}$$

where

$$\begin{aligned}K_{abcd} \equiv {} & \perp R_{abcd} + 3\omega_{ab}\omega_{cd} + \tfrac{2}{9}\theta^2 h_{b[c}h_{d]a} - 2(h_{a[d}\perp\phi_{\|c]b} + h_{b[c}\perp\phi_{\|d]a}) \\ & + 2(h_{a[d}\perp\phi_{|c]}\perp\phi_{|b} + h_{b[c}\perp\phi_{|d]}\perp\phi_{|a} - h_{b[c}h_{d]a}h^{sk}\phi_{|s}\phi_{|k}).\end{aligned} \tag{3.6}$$

Written explicitly in terms of the expansion term θ instead of ϕ we have the condition

$$
\begin{aligned}
&(\pounds - \tfrac{2}{3}\theta)(\perp R_{abcd} + 3\omega_{ab}\omega_{cd}) \\
&\quad = -\tfrac{2}{3}\perp\theta_{\|a[d}h_{c]b} - \tfrac{2}{3}\perp\theta_{\|b[c}h_{d]a} - \tfrac{4}{3}\perp\theta_{\|a}\alpha_{[d}h_{c]b} - \tfrac{4}{3}\perp\theta_{\|b}\alpha_{[c}h_{d]a} \\
&\qquad - \tfrac{4}{3}\perp\theta_{\|[d}h_{c]b}\alpha_a - \tfrac{4}{3}\perp\theta_{\|[c}h_{d]a}\alpha_b - \tfrac{4}{9}\theta\dot{\theta}h_{a[d}h_{c]b} \\
&\qquad - \tfrac{2}{3}\theta\omega_{ak}\omega^{k}_{.[d}h_{c]b} - \tfrac{2}{3}\theta\omega_{bk}\omega^{k}_{.[c}h_{d]a} \\
&\qquad + \tfrac{2}{3}\theta u^s u^k R_{sak[d}h_{c]b} + \tfrac{2}{3}\theta u^s u^k R_{sbk[c}h_{d]a}.
\end{aligned} \tag{3.7}
$$

This equation reduces on putting $\theta = 0$, to one of the integrability conditions for rigid motion obtained by Pirani and Williams.

Application of lemma 3 to (3.5) leads to the integrability conditions

$$\pounds\perp\Delta_p(e^{2\phi}K_{abcd}) = 0, \tag{3.8}$$

and repeated applications may be used to obtain more integrability conditions

$$\pounds\perp\Delta_q\perp\Delta_p(e^{2\phi}K_{abcd}) = 0, \ldots, \text{etc.} \tag{3.9}$$

The integrability conditions (3.5), (3.8), (3.9) may be shown to be the requirement that the Riemann tensor for the three-dimensional space $\overset{*}{V}$ should satisfy the equations

$$\frac{\partial}{\partial x^4}\overset{*}{R}_{\alpha\beta\gamma\delta} \doteq 0, \qquad \frac{\partial}{\partial x^4}\overset{*}{\nabla}_\lambda\overset{*}{R}_{\alpha\beta\gamma\delta} \doteq 0,$$

and so on.

Further integrability conditions obtained from (3.5) by contraction are

$$\pounds K_{ad} = 0, \qquad K_{ad} \equiv g^{bc}K_{abdc}; \tag{3.10}$$

and

$$\pounds(e^{-2\phi}K) = 0, \qquad K \equiv g^{ad}K_{ad}. \tag{3.11}$$

In terms of θ these equations are

$$
\begin{aligned}
&\pounds(\perp R_{ad} + R_{asdk}u^s u^k - 3\omega_{ak}\omega^{k}_{.d}) \\
&\quad = \tfrac{1}{3}\perp\theta_{\|ad} + \tfrac{2}{3}\perp\theta_{\|(a}\alpha_{d)} + \tfrac{1}{3}\theta\omega_{ak}\omega^{k}_{.d} - \tfrac{1}{3}\theta R_{asdk}u^s u^k \\
&\qquad + (\tfrac{1}{3}h^{sk}\theta_{\|sk} + \tfrac{2}{3}\theta_{\|k}\alpha^k + \tfrac{4}{9}\theta\dot{\theta} - \tfrac{1}{3}\theta\omega_{sk}\omega^{sk} - \tfrac{1}{3}\theta R_{sk}u^s u^k)h_{ad},
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
&\pounds(R + 2R_{sk}u^s u^k + 3\omega_{sk}\omega^{sk}) \\
&\quad = \tfrac{4}{3}h^{sk}\theta_{\|sk} + \tfrac{8}{3}\theta_{\|k}\alpha^k + \tfrac{4}{3}\theta\dot{\theta} - \tfrac{10}{3}\theta\omega_{sk}\omega^{sk} - \tfrac{2}{3}\theta R - \tfrac{8}{3}\theta R_{sk}u^s u^k.
\end{aligned} \tag{3.13}
$$

They also reduce to the rigid case on putting $\theta = 0$. Lemma 3 may be applied to (3.10) and (3.11) to give further integrability conditions:

$$\pounds\perp\Delta_b K_{ad} = 0, \qquad \pounds\perp\Delta_c\perp\Delta_b K_{ad} = 0, \qquad \text{etc,} \tag{3.14}$$

and

$$\pounds\perp\Delta_b(e^{-2\phi}K) = 0, \qquad \pounds\perp\Delta_c\perp\Delta_b(e^{-2\phi}K) = 0, \qquad \text{etc.} \tag{3.15}$$

These equations may be shown to require that the tensors $\overset{*}{R}_{\alpha\beta}$, $\overset{*}{R}$ and their covariant derivatives with respect to $\overset{*}{\nabla}$ should be independent of the coordinate x^4.

We may conclude that the equations (3.5), (3.10) and (3.11) together with those obtained from them by repeated application of lemma 3 are the compatibility conditions to be satisfied by u_a and its covariant derivatives in order that a given space–time admit a shear-free motion with four-velocity u_a. They generalize the results obtained by Pirani and Williams (1962) to the case when $\theta \neq 0$, showing explicitly the additional terms depending on θ. Attempts to obtain explicitly a set of selected variables in terms of which further derivatives may be algebraically expressed have proved unsuccessful due to the presence of terms involving θ and its first and second derivatives in the integrability conditions (3.7), (3.12), (3.13).

4. Shear-free fluids

So far we have confined our attention to the *kinematics* of shear-free motion. We have not specified the nature of the continuous medium under consideration nor any field equations. We now apply the results of § 3 to the *dynamics* of a perfect fluid.

We therefore consider space–times which are required to satisfy Einstein's field equations:

$$R_{ab}-\tfrac{1}{2}Rg_{ab} = -\kappa T_{ab} \tag{4.1}$$

where the energy–momentum tensor T_{ab} is given in terms of the proper density μ, the pressure p and the four-velocity u_a by

$$T_{ab} = \mu u_a u_b + p h_{ab}. \tag{4.2}$$

We assume that the fluid has an equation of state $\mu = \mu(p)$.

The conservation law $T^{ab}{}_{\|b} = 0$, gives

$$\dot{\mu}+(\mu+p)\theta = 0 \tag{4.3}$$

and

$$\alpha_a+\frac{\perp p_{\|a}}{\mu+p} = 0. \tag{4.4}$$

In terms of the index F, defined by Lichnerowicz (1955) as

$$F(p) = \exp\int\frac{\mathrm{d}p}{\mu+p},$$

they may be written in the form

$$\pounds \ln F+\frac{1}{\hat{\mu}}\theta = 0 \tag{4.5}$$

and

$$\alpha_a+\perp(\ln F)_{\|a} = 0 \tag{4.6}$$

with

$$\hat{\mu} \equiv \frac{\mathrm{d}\mu}{\mathrm{d}p}.$$

On differentiating (4.4) we get

$$p_{\|ab}+\dot{p}_{\|b}u_a+\dot{p}u_{a\|b}+(\mu+p)_{\|b}\alpha_a+(\mu+p)\alpha_{a\|b} = 0. \tag{4.7}$$

If we impose the conditions of shear-free motion, the terms $u_{a\|b}$ and $\alpha_{a\|b}$ are given by (3.1) and (3.2). Taking spatial projection on both indices and antisymmetrizing we obtain the integrability conditions for $p_{\|a}$ which may be put in the form

$$£(F\omega_{ab}) = 0. \tag{4.8}$$

Thus for perfect fluids undergoing shear-free motion the tensor $F\omega_{ab}$ is constant along any given stream line.

If in (4.7) we project one index spatially and contract the other with the four-velocity, we obtain using (2.9) the result

$$£(\hat{\mu}\alpha_a) = \perp\theta_{\|a}+\theta\alpha_a. \tag{4.9}$$

We may remark that the scalar ϕ introduced in § 3 for shear-free motion may be shown to be determined by the energy density by the relation

$$\phi = \frac{1}{3}\int\frac{\mathrm{d}\mu}{\mu+p}.$$

4.1. Integrability conditions

Application of lemma 3 of § 2 to the integrability condition $£(F\omega_{ab}) = 0$, yields further integrability conditions

$$£\perp\Delta_c(F\omega_{ab}) = 0. \tag{4.10}$$

Others may similarly be obtained by repeated application of the lemma. These integrability conditions together with those obtained in § 3 enable us to determine a set of selected variables in which to express the compatibility conditions for shear-free motion subject to the field equations (4.1) and (4.2).

From (2.10) and (4.8), equation (3.3) becomes

$$\begin{aligned}\omega_{ab\|c} = {}& \left(\frac{2}{3}-\frac{1}{\hat{\mu}}\right)\theta\omega_{ab}u_c+2u_{[a}\omega_{b]d}\alpha^d u_c-\tfrac{2}{3}\perp\theta_{\|[a}h_{b]c}-\omega_{ab}\alpha_c\\ &+2\alpha_{[a}\omega_{b]c}-u_{[a}\omega_{b]d}\omega^d_{\cdot c}-\perp R_{abcd}u^d-\tfrac{2}{3}\theta u_{[a}\omega_{b]c}.\end{aligned} \tag{4.11}$$

Similarly from (2.9) and (4.9), equations (3.2) imply

$$\begin{aligned}\alpha_{b\|c} = {}& \frac{\theta}{\hat{\mu}}\omega_{bc}-\left[\frac{1}{\hat{\mu}}\perp\theta_{\|b}+\left(\frac{1}{\hat{\mu}}+\frac{\hat{\hat{\mu}}}{\hat{\mu}^2}(\mu+p)\right)\theta\alpha_b\right]u_c+\tfrac{1}{3}\dot{\theta}h_{bc}\\ &-2u_{(b}\omega_{c)d}\alpha^d+\tfrac{2}{3}\theta\alpha_{(b}u_{c)}-\alpha_b\alpha_c+\omega_{bd}\omega^d_{\cdot c}\\ &-\alpha_d\alpha^d u_b u_c+\tfrac{1}{9}\theta^2 h_{bc}-R_{bacd}u^a u^d.\end{aligned} \tag{4.12}$$

We have succeeded, in equations (4.11) and (4.12) in expressing $\omega_{ab\|c}$ and $\alpha_{b\|c}$ essentially in terms of a set of variables

$$\{\mathrm{sv}\}_1 = \{u_a, \alpha_a, \omega_{ab}, \theta, \dot{\theta}, \perp\theta_{\|a}, p\}$$

together with the geometric quantities g_{ab}, R_{abcd}, $R_{abcd\|e}, \ldots$ and so on, which must satisfy the field equations. We may remark that since μ is a specified function of p, μ and all its derivatives are determined once p is known.

From the field equations we see that

$$R_{bd}u^b u^d = -\tfrac{1}{2}\kappa(\mu+3p) \qquad \text{and} \qquad R = \kappa(3p-\mu) \tag{4.13}$$

so that the scalar K in (3.11) may be written as

$$K = -2\{\kappa\mu+\tfrac{2}{3}\hat{\mu}\dot{\theta}+(\tfrac{2}{9}\hat{\mu}-\tfrac{1}{3})\theta^2+\tfrac{1}{3}\hat{\mu}(\mu+3p)-(\tfrac{2}{3}\hat{\mu}+\tfrac{3}{2})\omega_{bc}\omega^{bc}$$
$$-[\tfrac{2}{3}\hat{\mu}+\tfrac{1}{9}\hat{\mu}^2+\tfrac{2}{3}\hat{\hat{\mu}}(\mu+p)]\alpha_b\alpha^b\}. \tag{4.14}$$

It is then possible, using (3.11), to express $\ddot{\theta}$ in terms of $\{\mathrm{sv}\}_1$:

$$\ddot{\theta} = f_1\kappa\theta+f_2\theta\dot{\theta}+f_3\alpha^c\perp\theta_{\|c}+f_4\theta^3+f_5\theta\omega_{bc}\omega^{bc}+f_6\theta\alpha_c\alpha^c \tag{4.15}$$

where the coefficients $f_1,\ldots,f_6$ are functions of p, μ and derivatives of μ.

One of the integrability conditions for shear-free fluids that may be obtained from (4.10) is

$$£\{\mathrm{e}^{-2\phi}h^{bc}\perp\Delta_c(F\omega_{ab})\} = 0, \tag{4.16}$$

which on using the field equations reduces to

$$£\{\mathrm{e}^{-2\phi}F[(\tfrac{1}{3}\hat{\mu}-3)\omega_{ab}\alpha^b-\tfrac{2}{3}\perp\theta_{\|a}]\} = 0$$

and this enables us to write

$$\dot{\theta}_{\|a} = \left(\frac{1}{2}-\frac{9}{2\hat{\mu}}\right)\omega_{a.}{}^{b}(\perp\theta_{\|b}+\theta\alpha_b)+\frac{9}{2}£(\ln\hat{\mu})\omega_{ab}\alpha^b+\left(\frac{1}{\hat{\mu}}-\frac{2}{3}\right)\theta\perp\theta_{\|a}-\dot{\theta}\alpha_a-\ddot{\theta}u_a \tag{4.17}$$

where $\ddot{\theta}$ is given by (4.15) and $£(\ln\hat{\mu})$ may be expressed in terms of θ, p, μ and derivatives of μ. The formula (4.17) therefore expresses $\dot{\theta}_{\|a}$ in terms of $\{\mathrm{sv}\}_1$.

The integrability conditions

$$£\perp\Delta_a(\mathrm{e}^{-2\phi}K) = 0$$

in (3.15) may be computed explicitly with the aid of the field equations. They can then be expressed in the compact form

$$A_a{}^b\perp\theta_{\|b}+B_{ab}\alpha^b+C_{abc}\omega^{bc} = 0 \tag{4.18}$$

where the tensors A, B and C are functions of the set $\{\mathrm{sv}\}_2 \equiv \{u_a, \alpha_a, \omega_{ab}, \theta, \dot{\theta}, p\}$ together with the geometric quantities of the metric. We may therefore in principle solve (4.18) for $\perp\theta_{\|a}$ in terms of the new set of variables $\{\mathrm{sv}\}_2$.

The tensor $B_{ab}\alpha^b$ contains a term in $\alpha^c£\perp R_{cbad}u^b u^d$. This may be expressed explicitly in terms of $\{\mathrm{sv}\}_2$ using the field equations and the integrability condition $£K_{bc} = 0$. Also $C_{abc}\omega^{bc}$ has a term $\omega^{bc}£\perp R_{bcad}u^d$. To express this in terms of $\{\mathrm{sv}\}_2$ we observe that the shear-free condition and the integrability condition $£(F\omega_{ab}) = 0$ imply

$$£(\mathrm{e}^{-4\phi}F^2\omega_{bc}\omega^{bc}) = 0$$

and hence

$$£\perp\Delta_a(\mathrm{e}^{-4\phi}F^2\omega_{bc}\omega^{bc}) = 0.$$

These two equations together with the integrability condition $£\perp\Delta_a(F\omega_{bc}) = 0$ enable us to write $\omega^{bc}£\perp R_{bcad}u^d$ in terms of $\{\mathrm{sv}\}_2$.

We see therefore that the variables $\{\mathrm{sv}\}_2$ enable us to express the compatibility conditions for shear-free motion in the desired form. In other words it is now possible in principle to express further derivatives of $\{\mathrm{sv}\}_2$ algebraically in terms of $\{\mathrm{sv}\}_2$.

5. Discussion

The projection tensor is Lie transferred along the world lines of particles under rigid motion. As a result the local rest spaces of particles may be associated with a three-dimensional space which is in fact the quotient space of space–time over the world lines. Our analysis shows that for shear-free motion, space–time is not in general the direct product of a world line and a three-space. However we find that there is an associated riemannian space conformal to space–time which now has a quotient space $\overset{*}{V}$ and the metric of this three-space is found to differ from the projection tensor by a scalar factor.

The integrability conditions for shear-free motion obtained in this paper may be seen from this point of view as requiring that all geometric objects constructed using the metric of the quotient space $\overset{*}{V}$ should be three dimensional.

In the search for selected variables, we are not able to carry out the procedure followed by Pirani and Williams when we restrict ourselves to the kinematics of the motion. This is due to the presence of terms in the variable θ with derivatives up to the second order. However on applying the results to the dynamics of a perfect fluid, the conservation law for the energy–momentum provides additional integrability conditions which make it possible to carry out the program of obtaining explicitly a set of selected variables.

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